Problema 1

ΔABC ; $[AD$ - bisectoare, $D \in BC$

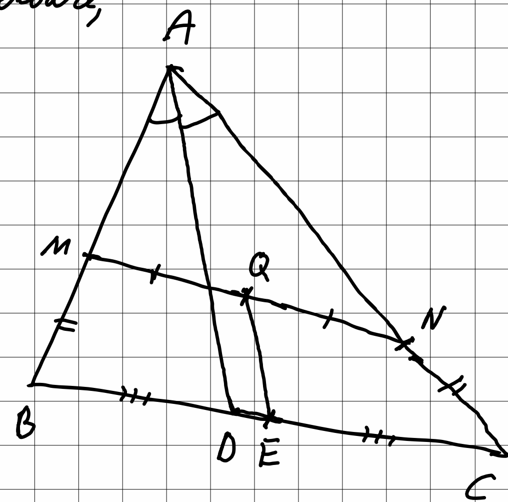
$M \in [AB]$

$N \in [AC]$

$BM = CN$

Q - mij. $[MN]$

E - mij. $[BC]$



$QE \parallel AD$

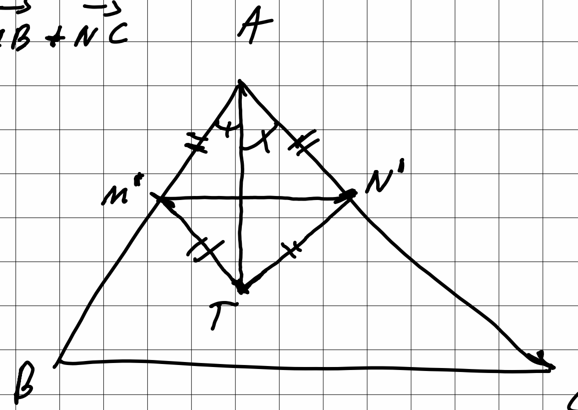
$$2\vec{QE} = \vec{QB} + \vec{QC}$$

$$= \vec{QM} + \vec{MB} + \vec{QN} + \vec{NC}$$

$$= \underbrace{\vec{QM} + \vec{QN}}_{=0} + \vec{MB} + \vec{NC}$$

$$= \vec{MB} + \vec{NC}$$

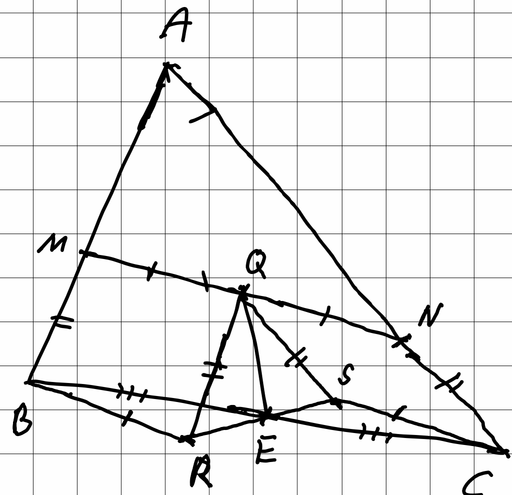
Translatăm $\vec{MB}$ și $\vec{NC}$ a.î. să aibă originea în A



$\vec{AM'} = \vec{MB}$ $\qquad \vec{AT} = \vec{AM'} + \vec{AN'}$

$\vec{AN'} = NC$

Soluție geometrică:



Fie R, S a.î. $MQRB$, $QNCS$ paralelog $\Rightarrow$

$\Rightarrow BR \underset{=}{\parallel} MQ \underset{=}{\parallel} QN \underset{=}{\parallel} SC \Rightarrow$

paralel și congruent

$\Rightarrow BRCS$ - paralelog $\Rightarrow E$ - mij $[RS]$

$QR \equiv MB \equiv NC \equiv QS$

$\Rightarrow (QE$ - bis. $\sphericalangle QRS$

$QR \parallel AB$; $QS \parallel AC$

$\Rightarrow QE \parallel AD$

Produs scalar

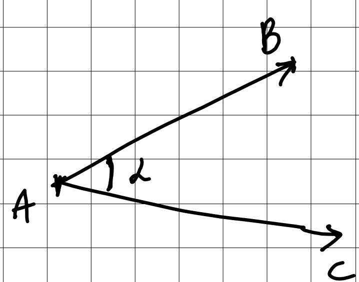
Fie $\overrightarrow{AB}$, $\overrightarrow{AC}$ doi vectori.

Definim $\overrightarrow{AB} \cdot \overrightarrow{AC} = \|\overrightarrow{AB}\| \cdot \|\overrightarrow{AC}\| \cdot \cos(\overrightarrow{AB}, \overrightarrow{AC})$

$\|\overrightarrow{AB}\|$ = lungimea seg. AB

$= AB$

$$\overrightarrow{AB} \cdot \overrightarrow{AC} = AB \cdot AC \cdot \cos(\overrightarrow{AB}, \overrightarrow{AC})$$



Proprietăți:

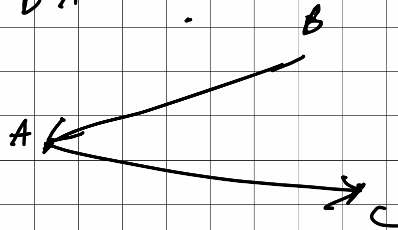
$$\overrightarrow{AB} \cdot \overrightarrow{AB} = AB^2$$

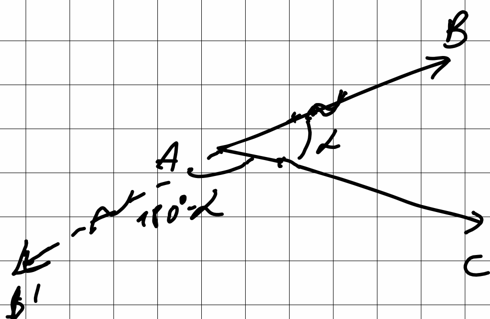
$$AB \perp AC \Leftrightarrow \overrightarrow{AB} \cdot \overrightarrow{AC} = 0$$

$|\overrightarrow{AB} \cdot \overrightarrow{AC}| \leq AB \cdot AC$ cu egalitate $\Leftrightarrow$ $\overrightarrow{AB}$, $\overrightarrow{AC}$ coliniari

$$\overrightarrow{AB} \cdot \overrightarrow{AC} = \overrightarrow{AC} \cdot \overrightarrow{AB}$$

$$\vec{AC}\cdot\vec{BA} = ?$$





$$\vec{AB'} = \vec{BA}$$

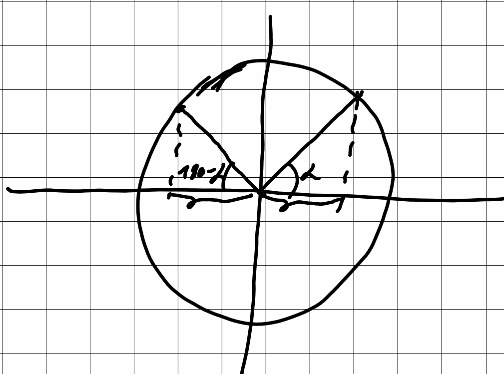
$$\vec{AC}\cdot\vec{BA} = \vec{AC}\cdot\vec{AB'} =$$

$$= AC\cdot AB'\cdot\cos(180°-\alpha)$$

$$= AC\cdot AB\cdot\cos(180°-\alpha)$$

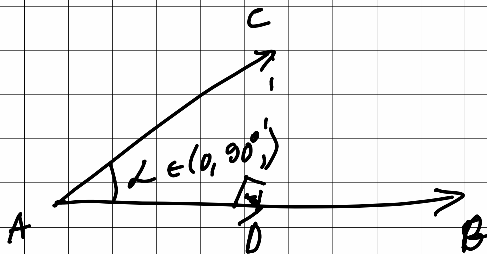
$$= -AC\cdot AB\cdot\cos\alpha$$

$$= -\vec{AC}\cdot\vec{AB}$$

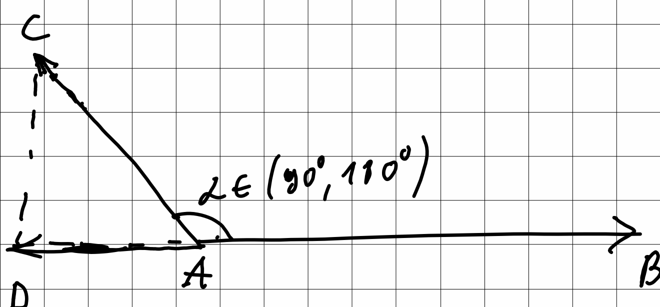


Concluzie: $\vec{AC}\cdot(-\vec{AB}) = -\vec{AC}\cdot\vec{AB}$

$\vec{AB}\cdot\vec{0} = \vec{0}$

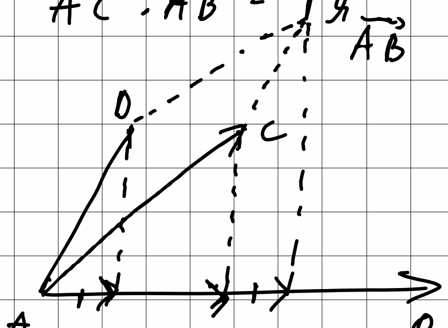


$$\vec{AB} \cdot \vec{AC} = AC \cdot AD$$



$$\vec{AC} \cdot \vec{AB} = -AC \cdot AD$$

Deci $\vec{AC} \cdot \vec{AB} = \overrightarrow{Pr_{\vec{AB}} \vec{AC}} \cdot \vec{AB}$

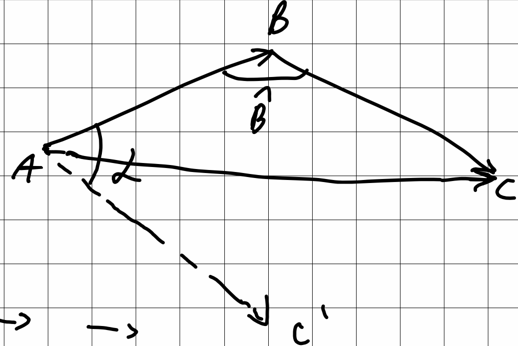


$$\overrightarrow{Pr_{\vec{AB}} (\vec{AC} + \vec{AD})} = \overrightarrow{Pr_{\vec{AB}} \vec{AC}} + \overrightarrow{Pr_{\vec{AB}} \vec{AD}}$$

De mai sus, $\vec{AB}\cdot(\vec{AC}+\vec{AD}) =$

$$= \vec{AB}\cdot\vec{AC} + \vec{AB}\cdot\vec{AD}.$$

Teorema cosinusului



$$\vec{AC} = \vec{AB} + \vec{BC}$$

$$AC^2 = \vec{AC}\cdot\vec{AC} = (\vec{AB}+\vec{BC})\cdot(\vec{AB}+\vec{BC})$$

$$= AB^2 + BC^2 + 2\,\vec{AB}\cdot\vec{BC}$$

Fie $\vec{AC'} = \vec{BC}$

$\alpha = 180^\circ - \hat{B}$ $\Rightarrow$ $\vec{AB}\cdot\vec{BC} =$

$$= \vec{AB}\cdot\vec{AC'} = AB\cdot AC'\cdot\cos\alpha$$

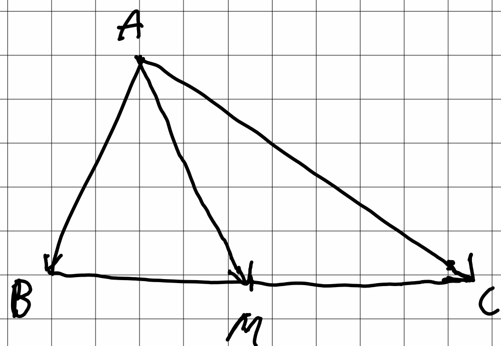
$$= AB\cdot BC\cdot\cos(180^\circ - \hat{B})$$

$$= -AB\cdot BC\cdot\cos(\hat{B})$$

$$AC^2 = AB^2 + BC^2 - 2\,AB \cdot BC \cdot \cos \hat{B}$$

$\square$

Lungimea medianei:



$$\vec{AM} = \frac{\vec{AB} + \vec{AC}}{2}$$

$$AM^2 = \vec{AM} \cdot \vec{AM} = \frac{1}{4} \cdot \left(AB^2 + AC^2 + 2\, \vec{AB} \cdot \vec{AC} \right)$$

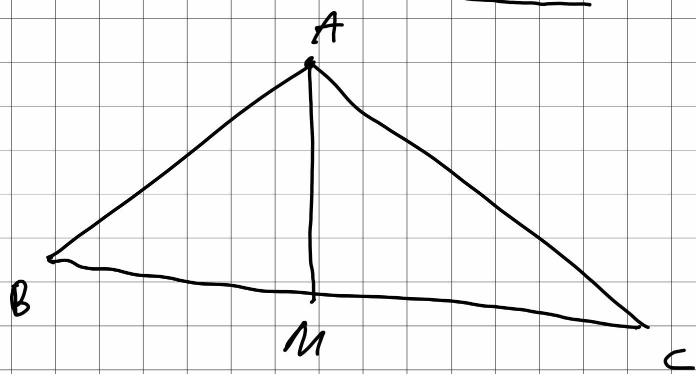
$$\vec{AB} \cdot \vec{AC} = AB \cdot AC \cdot \cos \hat{A}$$

$$\cos \hat{A} \overset{T.\cos.}{=} \frac{AB^2 + AC^2 - BC^2}{2\,AB \cdot AC}$$

$$2\,\vec{AB} \cdot \vec{AC} = AB^2 + AC^2 - BC^2 .$$

$$\text{Deci } AM^2 = \frac{1}{4}\left(2(AB^2 + AC^2) - BC^2\right) \quad \square$$

<u>Lungimea unei ceviene:</u>



$$\frac{BM}{MC} = t$$

$$\vec{AM} = \vec{AB} + \overbrace{\frac{t}{t+1} \cdot \vec{BC}}^{\vec{BM}}$$

$$\vec{AM} = \vec{AB} + \frac{t}{t+1} \cdot \left(\vec{AC} - \vec{AB}\right)$$

$$= \vec{AB} \cdot \frac{1}{t+1} + \frac{t}{t+1} \cdot \vec{AC}$$

$$AM^2 = \vec{AM} \cdot \vec{AM}$$

$$= \frac{1}{(t+1)^2} \cdot AB^2 + \frac{t^2}{(t+1)^2} \cdot AC^2 + \frac{2t}{(t+1)^2} \cdot \vec{AB} \cdot \vec{AC}$$

$$= \frac{AB^2 + t^2 AC^2 + t \cdot (AB^2 + AC^2 - BC^2)}{(t+1)^2}$$

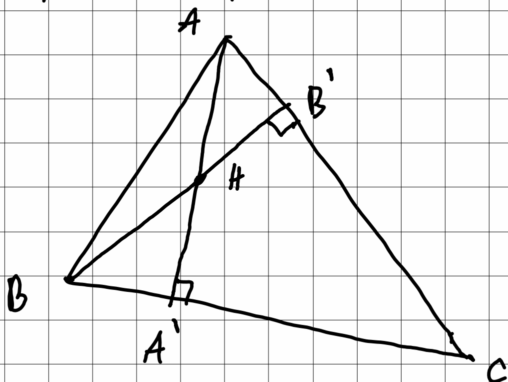
$\boxtimes$

Corolar (lungimea bisectoarei)

$$t = \frac{AB}{AC}$$

$\boxtimes$

Concurența înălțimilor



$$\overrightarrow{AH} \cdot \overrightarrow{BC} = 0$$

$$\overrightarrow{BH} \cdot \overrightarrow{AC} = 0$$

$$\overrightarrow{CH} \cdot \overrightarrow{AB} = 0$$

$$\overrightarrow{CH} = \overrightarrow{CA} + \overrightarrow{AH}$$

$$\overrightarrow{CH} \cdot \overrightarrow{AB} = (\overrightarrow{CA} + \overrightarrow{AH}) \cdot \overrightarrow{AB}$$

$$= \overrightarrow{CA} \cdot \overrightarrow{AB} + \overrightarrow{AH} \cdot \overrightarrow{AB}$$

$$= \overrightarrow{CA} \cdot \overrightarrow{AB} + \overrightarrow{AH} \cdot (\overrightarrow{AC} + \overrightarrow{CB})$$

$$= \overrightarrow{CA} \cdot \overrightarrow{AB} + \overrightarrow{AH} \cdot \overrightarrow{AC} + \underbrace{\overrightarrow{AH} \cdot \overrightarrow{CB}}_{=0}$$

$$= \overrightarrow{AC} \cdot (\overrightarrow{BA} + \overrightarrow{AH})$$

$$= \overrightarrow{AC} \cdot \overrightarrow{BH} = 0$$

Vectori în probl. de loc geometric.

Fie ABC un triunghi și $k \geq 0$

Să se det. locul geometric al punctelor M din plan a.î.

$$MA^2 + MB^2 + MC^2 = k \qquad + \text{condiții pt. } k$$

Sol:

Fie P fixat.

$$\left.\begin{aligned} \overrightarrow{MA} &= \overrightarrow{MP} + \overrightarrow{PA} \\ \overrightarrow{MB} &= \overrightarrow{MP} + \overrightarrow{PB} \\ \overrightarrow{MC} &= \overrightarrow{MP} + \overrightarrow{PC} \end{aligned}\right\}$$

e util să avem doar un vector variabil $\overrightarrow{MP}$

$$MA^2 = MP^2 + PA^2 + 2\,\overrightarrow{MP}\cdot\overrightarrow{PA}$$

și celelalte $+$

$$k = 3MP^2 + PA^2 + PB^2 + PC^2 + \underbrace{2\,\overrightarrow{MP}(\overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC})}_{\text{mă deranjează, ia să fie } 0.}$$

Aleg $P = G$

$$k = 3MG^2 + (GA^2 + GB^2 + GC^2)$$

$$MG^2 = \frac{k - GA^2 + GB^2 + GC^2}{3}$$ ⟸ echivalentă cu $MA^2 + MB^2 + MC^2 = k$

Condiție:

$$k \geq GA^2 + GB^2 + GC^2$$

L.g: Cerc cu centrul în G și rază

$$\sqrt{\frac{k - GA^2 + GB^2 + GC^2}{3}}.$$

Fie $\vec{v_1}, \vec{v_2}, \dots \vec{v_n}$ vectori în plan de lungime 1. S. s. a. că $(\exists)\ \varepsilon_1, \varepsilon_2, \dots \varepsilon_n = \pm 1$

a. î. $$\left\| \varepsilon_1 \vec{v_1} + \varepsilon_2 \vec{v_2} + \dots + \varepsilon_n \vec{v_n} \right\| \leq \sqrt{2}$$

Demonstrăm o afirmație mai tare cea pentru $\|\vec{v_i}\| \leq 1$, $(\forall)\ i=\overline{1,n}$.

Inducție după n.

Pt. $n=1$ clar

Pt. $n=2$, iau semnele a.î.

$$\sphericalangle(\varepsilon_1 v_1, \varepsilon_2 \vec{v_2}) \geq 90^\circ$$

$$\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + 2\vec{u}\vec{v} + \|\vec{v}\|^2$$

$$= \|\vec{u}\|^2 + \|\vec{v}\|^2 + 2\|\vec{u}\|\cdot\|\vec{v}\|\cdot\cos(\vec{u},\vec{v})$$

Dacă $\sphericalangle(\vec{u},\vec{v}) \geq 90^\circ \Rightarrow \cos(\vec{u},\vec{v}) \leq 0 \Rightarrow$

$$\Rightarrow \|\vec{u}+\vec{v}\|^2 \leq \|\vec{u}\|^2 + \|\vec{v}\|^2.$$

În cazul $\left.\begin{array}{l}\vec{u} = \varepsilon_1 \vec{v_1} \\ \vec{v} = \varepsilon_2 \vec{v_2}\end{array}\right\} \Rightarrow \|\varepsilon_1 \vec{v_1} + \varepsilon_2 \vec{v_2}\| \leq$

$$\leq \|\varepsilon_1 \vec{v_1}\| + \|\varepsilon_2 \vec{v_2}\| \leq 2.$$

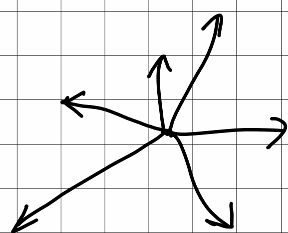
Pasul de inducție:

Presupunem că pt. orice n vectori cu $\|\cdot\| \le 1$ enunțul merge, , $n \ge 2$

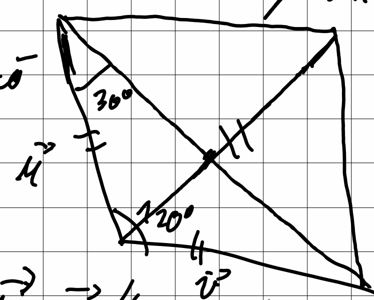
Fie $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{n+1}$ cu

$$\|\vec{v}_i\| \le 1, \quad i = \overline{1, n+1}$$

$\pm \vec{v}_1, \pm \vec{v}_2, \pm \vec{v}_3$ sunt 6 vectori în plan.



Dacă



$$\|\vec{u} + \vec{v}\| = \|\vec{u}\| + \|\vec{v}\|$$

Există 2 dintre ei, $\vec{u}$ și $\vec{v}$ a.î. $\sphericalangle(\vec{u}, \vec{v}) < 60°$.

Atunci $\sphericalangle(\vec{u}, -\vec{v}) > 120° \Rightarrow \cos(\vec{u}, \vec{v}) < -\frac{1}{2}$

$$\|\vec{u} - \vec{v}\|^2 = \|\vec{u}\|^2 + \|v\|^2 + 2\|\vec{u}\| \cdot \|\vec{v}\| \cdot \cos(\vec{u}, -\vec{v})$$

$$\|\vec{u} - \vec{v}\|^2 \le \|\vec{u}\|^2 + \|v\|^2 - \|\vec{u}\| \cdot \|\vec{v}\| \le 1,$$

căci $(\|\vec{u}\| - 1)(\|\vec{v}\| - 1) \ge 0$

Presupunem că $u = \pm v_1$

$v = \pm v_2$

Notăm $w = u - v$

$$\underbrace{\|w\|, \|\pm v_3\|, \|\pm v_4\|, \ldots, \|\pm v_{n+1}\|}_{n \text{ vectori.}} \leq 1$$

Din ip. de inducție $\Rightarrow$ $\boxtimes$

$$OG^2 = R^2 - \frac{1}{9}(a^2 + b^2 + c^2)$$

$$\vec{0} = \vec{GA} + \vec{GB} + \vec{GC} \Rightarrow$$

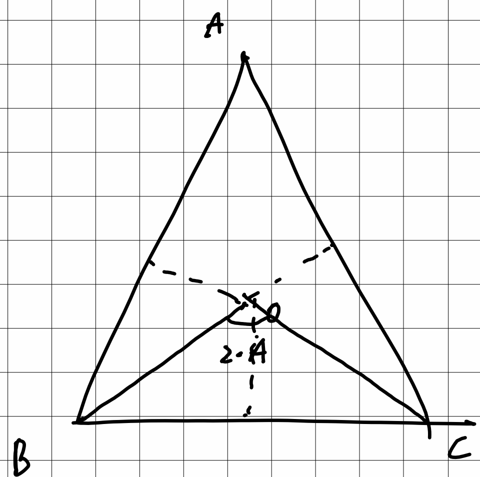
$$\Rightarrow \vec{0} = 3\vec{GO} + \vec{OA} + \vec{OB} + \vec{OC}$$

$$3\vec{OG} = \vec{OA} + \vec{OB} + \vec{OC}$$

$$9OG^2 = (\vec{OA} + \vec{OB} + \vec{OC})^2 \Leftarrow \text{prod. scalar}$$

$$(\vec{OA}+\vec{OB}+\vec{OC})^2 =$$

$$= OA^2 + OB^2 + OC^2 + 2(\vec{OA}\cdot\vec{OB} + \vec{OB}\cdot\vec{OC} + \vec{OC}\cdot\vec{OA})$$



$$\vec{OB}\cdot\vec{OC} = R\cdot R\cdot\cos(2\cdot A)$$

$$= R^2\cdot(\cos^2 A - \sin^2 A)$$

$$= R^2\cdot(1 - 2\sin^2 A)$$

$$= R^2 - \frac{BC^2}{2} \qquad \boxed{2R\sin A = BC}$$

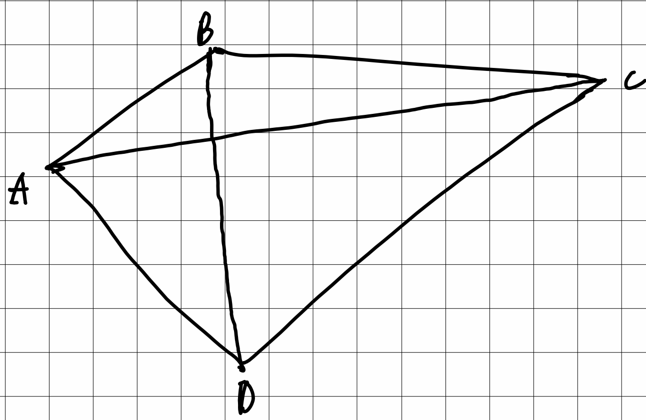
$$= R^2 - \frac{1}{2}a^2$$

$$9\,OG^2 = 3R^2 + 2\cdot\left(3R^2 - \frac{1}{2}\cdot(a^2+b^2+c^2)\right)$$

$$= 9R^2 - (a^2+b^2+c^2) \quad \square$$

Într-un patrulater $ABCD$

$$\vec{AC} \cdot \vec{BD} = \vec{AB} \cdot \vec{CD} + \vec{BC} \cdot \vec{AD}$$



$$\vec{AC} \cdot \vec{BD} = (\vec{AB} + \vec{BC}) \cdot (\vec{BC} + \vec{CD})$$

$$= \vec{AB} \cdot \vec{BC} + \vec{BC} \cdot \vec{BC} + \vec{AB} \cdot \vec{CD} +$$

$$+ \vec{BC} \cdot \vec{CD}$$

$$= \vec{AB} \cdot \vec{CD} + \vec{BC} \cdot \underbrace{(\vec{AB} + \vec{BC} + \vec{CD})}_{= \vec{AD}}$$